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On $1/N$ corrections to the entropy of noncommutative Yang-Mills theories

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Abstract: We study thermodynamical aspects of string theory in the limit in which it corresponds to Noncommutative Yang-Mills. We confirm, using the AdS/CFT correspondence, that for general Dp-branes the entropy in the planar approximation depends neither on the value of the background magnetic field B nor on its rank. We find $1/N^2$ corrections to the planar entropy in the WKB approximation. For all appropriate values of p these corrections are much softer than the corresponding corrections for the $B = 0$ case, and vanish altogether in the high temperature limit.

Keywords: 1/N Expansion, Lattice Gauge Field Theories, D-branes, Black Holes in String Theory.

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Contents

1. Introduction

In string theory, the perturbative two-dimensional world-sheet data contains information about the target-space geometry background on which the string propagates. Perturbatively one expects that an effective target-space theory will have on its world volume a definite geometry, the very same as that encoded in the two-dimensional world-sheet theory. However, one knows from T-duality that the metric on that world-volume may actually be ambiguous [1]. Moreover there is no *a priori* reason why new ambiguities may not emerge also non-perturbatively. Actually in some topological world-sheet theories the topology of the target space in the sigma-model does not define unambiguously the topology of the world-volume in target space [2]. In other cases such as type IIA string theory the world-volume dimensionality of the effective target-space theory is larger than ten and is actually eleven [3].

Recently [4, 5] it was found that the world-volume of the effective target-space theory may be smaller than that expected from the two-dimensional sigma-model, moreover the effective target-space theory may be just a field theory not containing gravity. For example $\mathcal{N} = 4$, $D = 4$, supersymmetric Yang–Mills (SYM) describes a string propagating in the ten-dimensional target space $AdS_5 \times S^5$. Not only are the geometry, the dimensionality and topology of the target-space manifolds largely modified but also the commutativity properties of the target-space coordinates were found to be ambiguous: in a certain limit [6], in which the background field B is non-vanishing in some directions, the metric components are of $O(\epsilon)$ and the string Regge slope α' is of order $O(\sqrt{\epsilon})$, a string probe will not distinguish between one gauge system on a target-space manifold whose variables are commutative and another gauge system on a noncommutative (NC) target space whose coordinates satisfy [7]

$$[x^i, x^j] = i\,\theta^{ij}\,, \tag{1.1}$$

with $\theta \sim B^{-1}$. This limit was originally introduced [8] as a limit of M-theory in the context of the matrix model of ref. [9]. The weak-coupling picture in terms of standard open strings on D-branes was presented in [10] (see also [11]) and further studied from various points of view in [12].

For many years one has been curious about the manner in which such non-commutativity would influence the behaviour of the system. Given that in string theory there are equivalences between theories on commutative and non commutative spaces it may have not been a total surprise that the entropies of both systems were found to be indentical in a certain large-N limit. In fact it was shown that, at the level of the large-N diagrammatic combinatorics, the modifications implied by the presence of the torsion can be relegated to torsion-dependent phase factors mutiplying torsionless n-point functions in momentum space [13, 14]. In particular zero-point functions such as the free energy and the entropy are unchanged. Although the close similarity of thermodynamical functions at low temperatures (compared to the noncommutativity scale) was expected, the fact that similarities persist, for large N, at high temperatures was more of a surprise. These identities have been demonstrated also by applying the AdS/CFT correspondence and considering the supergravity limit on the AdS side, a description valid at strong 't Hooft coupling $g^2_{\rm YM}N \gg 1$. The appropriate masterfield has been identified for the zero-temperature system [15, 16] and the additional masterfield, a black-hole configuration, has been identified for the finite-temperature case [16]. While these supergravity configurations do depend on the background torsion field, it has been shown by calculating classically the black-hole entropy that large-N thermodynamical functions are B-independent.

The classical supergravity results capture the leading N properties of the gauge-theory side. There are no known limitations, based on diagrammatic analysis, on the B-dependence of the entropy to next to leading order in $1/N^2$. It may not be so easy to estimate these corrections in the non-commutative gauge-theory language. However one may also approach the calculation from the supergravity side of the correspondence. We have used in ref. [17] a WKB approximation to estimate the next to leading order effects on the entropy for vanishing values of B. This method did not take into account stringy quantum corrections but did account for quantum corrections to the classical supergravity picture. This is actually usefull for the limit at hand in which stringy effects are decoupled.

As the main result of this paper, we find that the indroduction of the magnetic B-field, or in other variables a non-infinite θ background, leads to a large suppression of the next to leading order free energy and entropy, in fact the entropy correction vanishes at large temperatures, leaving only the leading order entropy. Equivalently, the limit of infinite NC length $\theta \to \infty$, at *fixed* energy is completely saturated by the planar limit. For vanishing θ, i.e. for the gauge theory on a commuting manifold the next to leading order results are of the same qualitative properties as the leading result.

In section 2 we review the WKB approximation to the $1/N$ corrections to the free energy and entropy. We apply the method to obtain highly suppressed $1/N$ corrections in the case of the four-dimensional gauge theory, i.e. the case of a large stack of N D3-branes. We also recover the commuting case results for a vanishing value of θ, that is in the limit of small and large B fields.

In section 3 these results are generalized to Dp branes (as well as NS5-branes). We study the entropy of the system as a function of both p and the rank r of the magnetic field B. For all $p < 7$ the large-temperature planar entropy of the black-hole configuration does not depend on the value of B. The more complicated behaviour for $7 > p > 4$ is also unchanged. The $1/N^2$ corrections do depend on the value of B. The main result is again that we find them to be always much softer than those corresponding to the $B = 0$ case. We finally touch upon the most general configuration which supposedly describes the supergravity side in the aditional presence of a constant electric field.

After this work has been completed we received the article [23], which contains some overlap with the material in section 3.

2. WKB estimations of $1/N$ corrections to the entropy

In the context of the AdS/CFT correspondence, the sum over diagrams with toroidal topology in the 't Hooft classification should be related, at strong coupling, to the one-loop diagrams of the corresponding dual string theory. In our case, we are interested in the one-loop (toroidal world-sheet) diagram of type-IIB string theory in the appropriate string background which leads to a gauge theory.

Lacking an operative description in terms of an exact CFT, the calculation of this diagram in closed form is beyond our capabilities. We can, however, produce estimates by means of approximations to its low-energy limit: the one-loop diagrams of IIB supergravity.

One observable which is rather independent of the regularization ambiguities of supergravity is the vacuum-subtracted statistical free energy. At one-loop, the supergravity fields can be regarded as free (interacting only with the background) and the thermal free energy can be evaluated by means of an oscillator sum for each field:

$$\beta\, F(\beta) = \sum_{\rm species} {\rm Tr}(-1)^{\rm F} \log\left(1-(-1)^{\rm F}\, e^{-\beta\,\omega}\right), \tag{2.1}$$

where F is the space-time fermion number and the trace is over the spectrum of physical fluctuations in the background, with frequencies ω given by the eigenvalues of the operator $i\partial_t$, associated to one particular temporal Killing vector of the background, which we assume static. If we evaluate (2.1) by some determinant on the compactified euclidean continuation of the background, the inverse temperature β is the period of identification of the euclidean time.

The vacuum energy has been subtracted in this definition of the free energy, so that the relation to the one-loop path integral is

$$I^{(1)} = \beta\, E_{\rm vac} + \beta\, F(\beta)\,. \tag{2.2}$$

Under the assumption of *local extensivity* we can estimate the statistical sum by

$$\beta\, F(\beta) \approx \int {\rm d}^d x \sqrt{|g|}\, \beta_{\rm loc}\, F_0(\beta_{\rm loc})\,, \tag{2.3}$$

with a red-shifted local inverse temperature $\beta_{\rm local} = \beta\sqrt{g_{tt}}$, and F_0 the flat-space free energy. Namely, this is an adiabatic approximation in which one assumes local thermal equilibrium in cells which are still small compared to the global features of the geometry, so that their contribution to the total free energy is given by the red-shifted flat-space free energy of the cell, and one further assumes extensivity with respect to the partition in cells. These assumptions can be justified with a standard application of the WKB approximation to the solutions of the wave equation $\omega^2 + \partial_t^2 = 0$ (see for example refs. [18]). Thus, the WKB approximation is good if the metric is sufficiently smooth. For example, for AdS_{d+1} with radius R, the derivative of the inverse local temperature $|\partial_r \beta_{\rm loc}|$ is bounded by β/R, with β the temperature at the centre. According to the AdS/CFT correspondence, this ratio is small precisely in the high-temperature limit, and our WKB approximation is better the higher the temperature.

For massless fields in d space-time dimensions, one has $\beta F_0 = -A\, T^{d-1}$, with A a constant proportional to the total number of particle degrees of freedom. Thus, the WKB approximation involves integration of $(\sqrt{g_{tt}})^{1-d}$ over the volume. A more geometrical characterization can be given defining the *optical metric* [19] by the conformal transformation

$$ds^2_{\rm optical} = \frac{1}{g_{tt}}\, ds^2_{\rm euclidean}\,, \tag{2.4}$$

from the Wick-rotated metric with compact time $t \equiv t + \beta$. Then, the contribution of a given region $\mathbf{X}$ of space-time to the one-loop free energy is proportional to the optical volume of this region, which we shall denote by $\widetilde{\rm Vol}(\mathbf{X})$.

For space-times with a black hole, we should consider only the optical volume of the region not excluded by the black-hole horizon. The reason being that the euclidean metric of a black hole terminates at the horizon, which represents a radial cut-off. Strictly speaking, the red-shift estimate breaks down very close to the horizon, because the local temperature diverges. This local divergence can be interpreted as contributing to the renormalization of the Newton constant, as in ref. [20]. Thus, in dealing with black-hole spacetimes, we shall consider only the optical volume of the asymptotic region, sufficiently far from the horizon, i.e. we consider the free energy of the Hawking radiation in equilibrium with the black hole. This naive subtraction of the horizon divergence is enough for the purpose of order of magnitude estimates [17].

For a curved space-time, a field is regarded as massless if its mass is smaller than the local temperature $T_{\rm loc} = T/\sqrt{g_{tt}}$. Otherwise, it is massive, and can be decoupled from the statistical sum in that region of space-time. This means that, for a situation with locally varying Kaluza–Klein thresholds one may partition the whole manifold $\mathbf{X}$ in cells $\mathbf{X}_i$ of effective dimension d_i, defined by the condition that the effective radii be sufficiently large compared to the local temperature:

$$\beta_{\rm loc} \ll R_{\rm loc}\,. \tag{2.5}$$

Finally, neglecting threshold effects, from the regions where $\beta_{\rm loc} \approx R_{\rm loc}$, the WKB approximation to the one-loop free energy can be written:

$$I^{(1)}_{\rm WKB} = \beta\, E_{\rm vac} - \sum_i A_i\, T^{d_i}\, \widetilde{\rm Vol}(\mathbf{X}_i)\,, \tag{2.6}$$

for a decomposition in "cells" $\mathbf{X}_i$, each of effective naive dimension d_i.

If we can isolate a regime where the dominant asymptotics is of the form $\beta F \sim -A\, T^\gamma$, the corresponding one-loop entropy is

$$S^{(1)}_{\rm WKB} \sim (\gamma+1)\, A\, T^\gamma\,. \tag{2.7}$$

In this case, we can define

$$d_{\rm eff} = \gamma + 1 \tag{2.8}$$

to be the effective dimension, as determined by the high-temperature asymptotics. Notice that this dimension is in general different from the naive dimensions d_i of the cells in which we have partitioned the manifold. The reason is that the optical volume of a given cell may depend non-trivially on the asymptotic reference temperature. For example, for AdS_{d+1}, and generalizations involved in the AdS/CFT correspondence, the effective dimension as defined by the high-temperature asymptotics is d instead of $d+1$, [17, 21]. This is in fact a manifestation of holography at the level of $O(1/N^2)$ corrections.

2.1 The basic example

The simplest example is given by the gravitational description of the $\mathcal{N}=4$ SYM theory at large N, obtained from a stack of D3-branes with a non-zero B-field on a single spatial two-plane. At large 't Hooft coupling $\lambda = g^2_{\rm YM} N$, the master field of the theory with a NC parameter θ is encoded in the metric derived in refs. [15] and [16]:

$$\frac{ds^2}{\alpha'} = \frac{U^2}{\sqrt{\lambda}}\left(-dt^2 + dy^2 + \hat{f}(U)\, d\mathbf{x}^2\right) + \sqrt{\lambda}\left(\frac{dU^2}{U^2} + d\Omega_5^2\right), \tag{2.9}$$

with

$$\hat{f}(U) = \frac{1}{1+(U\Delta)^4} \tag{2.10}$$

and $\Delta = \lambda^{-1/4}\sqrt{\theta}$. Notice that the perturbative NC energy scale, $\theta^{-1/2}$, differs by powers of the 't Hooft coupling from the value of the U-coordinate threshold for the onset of NC effects in the metric (2.9), which is given by $U_\Delta = 1/\Delta$. The associated length scale in the gauge theory, according to the UV/IR correspondence [25], is given by

$$a = \frac{\sqrt{\lambda}}{U_\Delta} = \lambda^{1/4}\sqrt{\theta}\,, \tag{2.11}$$

it differs from the weak-coupling NC length scale, $\sqrt{\theta}$, by powers of the 't Hooft coupling.[1] The small U or infrared region is the standard $AdS_5 \times S^5$ space with radius $R = \sqrt{\alpha'}\,\lambda^{1/4}$, in agreement with expectations, since NC effects should be irrelevant in the deep infrared regime. Conversely, the $\theta \to 0$ limit at fixed energy and coupling gives back the standard large-N master field of the commutative theory.

In the non-extremal case the horizon sits at $U_0 = T\sqrt{\lambda}$. The local value of the inverse temperature is $\beta_{\rm loc} = \beta\, U\, R/\sqrt{\lambda}$ for $U \gg U_0$, while the local value of the S^5 radius is $R(S^5) = R$. So, for $U > U_0 = T\sqrt{\lambda}$ we drop the five-sphere and the effective (euclidean) optical metric of interest is

$$ds^2_{\rm optical} = dt^2 + dy^2 + \hat{f}(U)\, d\mathbf{x}^2 + \lambda\,\frac{dU^2}{U^4}\,, \tag{2.12}$$

with optical volume

$$\widetilde{\rm Vol} = \sqrt{\lambda}\int_{U_0}^{\infty} dt\, dy\, d\mathbf{x}\, dU\, \frac{\hat{f}(U)}{U^2}\,. \tag{2.13}$$

So we finally obtain

$$I^{(1)}_{\rm WKB} = \beta\, E_{\rm vac} - A\,(LT)^3\,\mathcal{I}(aT) \tag{2.14}$$

in terms of the integral

$$\mathcal{I}(aT) = \int_1^\infty \frac{dx}{x^2(1+(aT)^4 x^4)}\,, \tag{2.15}$$

which can be explicitly evaluated:

$$\mathcal{I} = 1 - \frac{\pi aT}{\sqrt{8}} + \frac{aT}{4\sqrt{2}}\Bigg[2\arctan(1+\sqrt{2}aT) - 2\arctan(1-\sqrt{2}aT) + \\ + \log\left(\frac{1-\sqrt{2}aT+(aT)^2}{1+\sqrt{2}aT+(aT)^2}\right)\Bigg]$$

[1]We have absorbed various $O(1)$ constants in the definition of λ and α'.

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The important feature of this function is that it represents a small correction at low temperature $aT \ll 1$:

$$\mathcal{I} = 1 - \frac{\pi a T}{\sqrt{8}} + \frac{(aT)^4}{3} - \frac{(aT)^8}{7} + \cdots , \tag{2.16}$$

but a large suppression for very high temperatures, compared to the NC scale $aT \gg 1$:

$$\mathcal{I} \longrightarrow \frac{1}{5(aT)^4} - \frac{1}{9(aT)^8} + \cdots \tag{2.17}$$

This means that the one-loop free energy scales like a vacuum contribution in the large temperature limit $aT \gg 1$. In other words, the $1/N^2$ corrections to entropy *vanish* in such a limit: the extra contribution from the extreme ultraviolet regime is as if it represented a zero-dimensional volume. As mentioned in the introduction, the planar $O(N^2)$ entropy is expected to be the same as the non-commutative one, on the grounds of weak-coupling arguments [14]. This fact was verified at strong-coupling in ref. [16], i.e. the horizon area in Einstein frame does not change beyond the NC scale a. Our result indicates that, at least to leading order, the source of all the high-temperature entropy is in the planar evaluation of degrees of freedom.

Conversely, we can say that, in the limit of large NC parameter $\theta \to \infty$, at fixed energy, we are led to a purely planar theory, in the sense that the large-N description is effectively classical (trivial $1/N$ corrections).

3. Generalizations

Most of the previous discussion admits generalization to general Dp-branes with $1 < p < 7$. The supergravity string-frame solution for a stack of N Dp-branes with an aligned B-field (before the decoupling limits) is given in ref. [16]:

$$ds^2 = \frac{1}{\sqrt{H(\rho)}} \left(-dt^2 + d\mathbf{y}^2 + f(\rho)\, d\mathbf{x}^2\right) + \sqrt{H(\rho)} \left(d\rho^2 + \rho^2 d\Omega_{8-p}^2\right) . \tag{3.1}$$

There is a B-field of rank $2r$ in the *spatial* directions $\mathbf{x}$ of intensity[2]

$$\alpha' \bar{B} = \tan\vartheta \, \frac{f(\rho)}{H(\rho)} \tag{3.2}$$

and dilaton

$$e^{2\phi} = e^{2\phi_\infty} H(\rho)^{\frac{3-p}{2}} f(\rho)^r . \tag{3.3}$$

The functions $H(\rho)$ and $f(\rho)$ are given by

$$H(\rho) = 1 + \left(\frac{R}{\rho}\right)^{7-p} , \qquad f(\rho)^{-1} = \sin^2\vartheta H(\rho)^{-1} + \cos^2\vartheta . \tag{3.4}$$

[2] We write here the value of the skew-eigenvalues, which we assume all equal in magnitude, for simplicity of notation.

The basic low-energy scaling of ref. [4] $\rho = \alpha' U$ leads to

$$H(U) \longrightarrow \frac{\lambda}{(\alpha')^2 U^{7-p}}\,, \tag{3.5}$$

where $R = \sqrt{\alpha'}\,(G_s N)^{\frac{1}{7-p}}$ in terms of the NC string coupling G_s and the corresponding 't Hooft coupling $\lambda = g_{\rm YM}^2 N = (\alpha')^{\frac{p-3}{2}}\,G_s N$, where we have again absorbed various constants of $O(1)$ into the definitions of the parameters.

The low-energy scaling introduced by Seiberg and Witten in ref. [6] shrinks the closed string metric in the direction of the $\mathbf{x}$ coordinates, with a constant B-field. We may achieve this in the previous solution by a rescaling of the coordinates $\mathbf{x} \to \alpha' \mathbf{x}/\theta$, in the $\alpha' \to 0$ limit with constant $\theta = \alpha' \tan\vartheta$. At the same time, the original string coupling is scaled $e^{\phi_\infty} \to G_s\,(\alpha'/\theta)^r$, and the B-field transforms like a normal tensor under the rescaling: $\bar{B} \to B \cdot (\alpha'/\theta)^2$. In terms of the convenient NC length scale Δ defined by

$$\Delta^{7-p} = \frac{\theta^2}{\lambda}\,, \tag{3.6}$$

we get the following string metric after this double Maldacena–Seiberg–Witten scaling:

$$\frac{ds^2}{\alpha'} = \frac{U^{\frac{7-p}{2}}}{\sqrt{\lambda}}\left(-dt^2 + d\mathbf{y}^2 + \hat{f}(U)\,d\mathbf{x}^2\right) + \frac{\sqrt{\lambda}}{U^{\frac{7-p}{2}}}\left(dU^2 + U^2 d\Omega_{8-p}^2\right), \tag{3.7}$$

with

$$\hat{f}(U) = \frac{1}{1 + (U\Delta)^{7-p}}\,. \tag{3.8}$$

This result agrees with the recent determination of this function in ref. [22]. The U-dependent B-field profile is

$$B = B_\infty\,(U\Delta)^{7-p}\,\hat{f}(U)\,, \tag{3.9}$$

with $B_\infty = 1/\theta$. This asymptotic value of the B-field agrees with the zero slope limit of ref. [6] for the NC parameter matrix:

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B_\infty}\right)_A^{ij} \tag{3.10}$$

with g_{ij} the closed-string metric. A potential confusion stems from the fact that the NC parameter in this formula vanishes *both* for large and small values of the B-field. On the other hand, if $B_\infty = 1/\theta$, the limit of vanishing B-field seems to make NC effects blow up. This is resolved by noticing that the NC parameter vanishes with the B-field *only* if α' and the closed-string metric are kept fixed, namely the two limits that turn-off θ do not commute. In the supergravity solution, keeping the open-string scale fixed (Born–Infeld corrections) amounts to keeping the "neck" of

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the throat at $U_s \sim (\lambda/(\alpha')^2)^{\frac{1}{7-p}}$ in place in the full solution (3.1). If we now take the vanishing B-field limit with constant g_{ij} and constant α', we find $f(\rho) \to 0$ and NC features vanish as it should be.

Coming back to the scaled solution, the dilaton is

$$e^{2\phi} = G_s^2 \hat{f}(U)^r H(U)^{\frac{3-p}{2}} = e^{2\phi_{\mathrm{C}}} \hat{f}(U)^r \,, \tag{3.11}$$

where ϕ_{C} denotes the dilaton of the $\Delta = 0$ theory.

With these data, one could study the interplay of phase transitions in these models, depending on the local duality transformations appropriate for each description. Compared to the analysis of the commutative case in ref. [24], the NC character introduces a new scale in the problem at $U_\Delta = 1/\Delta$, associated to the onset of NC effects. The corresponding *length scale* in the gauge theory, according to the generalized UV/IR correspondence of [25] is

$$a = \sqrt{\frac{\lambda}{U_\Delta^{5-p}}} = \sqrt{\lambda\, \Delta^{5-p}}\,. \tag{3.12}$$

Following [24], the applicability of the supergravity description is controlled by the size of α' corrections in the string-metric background. In terms of the "correspondence point" $U_c = \lambda^{\frac{1}{3-p}}$ of ref. [26], one finds that the geometric description is good for $U \ll U_c$ when $p < 3$. Therefore, we need $U_\Delta < U_c$ in order to trust the supergravity solution in the region where NC effects are sizeable. In terms of the 't Hooft coupling versus the gauge-theory NC length scale, this condition is $\lambda^{3-p}\, a < 1$, i.e. we require a sufficently *weak* coupling. Otherwise, the NC features of the ultraviolet regime must be studied entirely by means of perturbative techniques.

For $p = 3$, the condition for the supergravity picture to capture NC features in the ultraviolet is the ordinary one, independent of the scale: $\lambda > 1$.

On the other hand, for $p = 4$ the supergravity patch is $U \gg U_c = 1/\lambda$, so that the NC features are visible in the supergravity description for sufficiently *strong* coupling: $\lambda > a$.

Finally, the cases $p = 5, 6$ are somewhat different since they do not follow a standard IR/UV correspondence (equation (3.12) does not have a clear physical interpretation in these cases). Still, we can associate NC effects to the *energy* scale U_Δ, as measured for example by the mass of a stretched fundamental string probe. The condition for the metric (3.7) to accurately describe the NC effects is thus $\lambda\, (U_\Delta)^{p-3} > 1$.

The non-perturbative thresholds associated with large values of the string dilaton are generally relaxed by turning on the NC moduli. Since $\hat{f} \to U^{p-7}$ vanishes in the large U regime, this means that the present metrics have small local string coupling in the $U \to \infty$ region for *all* values of p, provided $r \geq 1$ (in fact, one needs the slightly stronger condition $r \geq 2$ for $p = 6$). Following the general rule, the infrared thresholds associated with small $U < U_\Delta$ singularities are qualitatively the same in the NC case.

In general, there could be intermediate regimes with large local string coupling, but such transients can be ignored when working in the 't Hooft limit with fixed values of the typical energies in the system, as well as λ and Δ, of $O(N^0)$.

3.1 Planar thermodynamics

The (somewhat surprising) robustness of the planar thermodynamics of the $p = 3$ case, discussed in [16], persists for general values of p. The non-extremal metric is obtained by replacing

$$-dt^2 \to +h\,dt^2, \qquad dU^2 \to \frac{dU^2}{h}\,, \tag{3.13}$$

with the euclidean time identified with period the inverse temperature $t \equiv t + \beta$ and

$$h = 1 - \left(\frac{U_0}{U}\right)^{7-p} \tag{3.14}$$

as usual. Since no B-field lies in the time direction, these replacements do not affect the parts of the metric which depend on Δ (the $\mathbf{x}$ space). Therefore, the NC Hawking temperature is the same as in the commutative case.

$$T_{\rm NC} = T_{\rm C} = \frac{7-p}{4\pi}\sqrt{\frac{U_0^{5-p}}{\lambda}}\,. \tag{3.15}$$

Moreover, the planar entropy is also independent of the NC deformation parameter. Since it must be computed in the Einstein frame, we have to multiply the string metric by $e^{-\phi_{\rm NC}/2} = e^{-\phi_{\rm C}/2}\hat{f}(U)^{-r/4}$. The horizon being eight-dimensional, this yields a factor of $(\hat{f}^{-r/8})^8$, which exactly cancels the extra factor of $(\hat{f}^{1/2})^{2r}$ coming from the $2r$ directions with a non-vanishing B-field. So, the NC horizon area is

$$A_{\rm NC} = A_{\rm C}\,(\hat{f}^{1/2})^{2r}\,(\hat{f}^{-r/8})^8 = A_{\rm C}\,. \tag{3.16}$$

Both the planar entropy and the temperature are exactly the same as in the commutative case, which means that all *planar* thermodynamical functions are the same.

3.2 WKB corrections to the entropy

In order to estimate the $1/N$ corrections, we consider the corresponding optical metric

$$ds^2_{\rm optical} = dt^2 + d\mathbf{y}^2 + \hat{f}(U)\,d\mathbf{x}^2 + \frac{\lambda}{U^{7-p}}\left(dU^2 + U^2 d\Omega^2_{8-p}\right), \tag{3.17}$$

and compute the optical volume of the region $U_0 < U < \infty$. The conditions for decoupling the angular sphere S^{8-p} are the same as in the commutative case, again because the NC character only affects the $\mathbf{x}$-space. We discuss the qualitatively different cases in turn.

Dp-branes with $p<5$. For $p<5$, the temperature is small: $\beta_{\rm loc} > R(S^{8-p})_{\rm loc}$ in the region of interest, so that we can drop the angular sphere in estimating the free energy of thermal radiation outside the black-brane.

$$\begin{aligned} I^{(1)}_{\rm WKB} &\sim -T^{p+2}\,\widetilde{\rm Vol}_{p+2} \\ &= -T^{p+1}\,L^p \int_{U_0}^\infty dU\,\hat{f}(U)^r\sqrt{\lambda U^{p-7}} \\ &\sim -(L\,T)^p \int_1^\infty \frac{dx\sqrt{x^{p-7}}}{(1+(xU_0\Delta)^{7-p})^r}\,. \end{aligned} \tag{3.18}$$

This is the standard result of the commutative theory $I^{(1)}_{\rm WKB} \sim -(L\,T)^p$ for $U_0\Delta \ll 1$. On the other hand, in the opposite limit $T a \equiv T\sqrt{\lambda\Delta^{5-p}} \gg 1$, we get a strong suppression

$$I^{(1)}_{\rm WKB} \longrightarrow -\frac{(L\,T)^p}{(a\,T)^{\frac{2r(7-p)}{5-p}}}\,. \tag{3.19}$$

Thus, we find the soft behaviour at high temperatures of the one-loop free energy, much like the D3-brane case. Notice that for all $p<5$, the asymptotic effective exponent of T is negative provided $r \geq 1$. Therefore, the effective dimensionality, as determined by the one-loop corrections, drops to zero or is even "negative" at $T a \gg 1$.

D5-branes. For $p=5$ one gets, independently of the issue of angular decoupling:

$$I^{(1)}_{\rm WKB} \sim -(L\,T)^5 \int_1^\infty \frac{dx}{x}\frac{1}{(1+(xU_0\Delta)^2)^r}\,. \tag{3.20}$$

As long as $r>0$, the integral converges! This is an improvement with respect to the commutative case, with $\Delta = 0$, in which one gets a logarithmic divergence of dubious interpretation. In the $U_0\Delta \gg 1$ regime one finds

$$I^{(1)}_{\rm WKB} \longrightarrow -\frac{(L\,T)^5}{(U_0\Delta)^{2r}}\,. \tag{3.21}$$

However, now the energy-density parameter U_0 is unrelated to the temperature, which is constant and equal to $\lambda^{-1/2}$, i.e. there is a suppression of the non-planar corrections, although the effective dimension remains $d_{\rm eff} = 6$.

D6-branes. On the other hand, for $p=6$, the local temperature outside the horizon is higher than the mass of angular modes and we must consider the optical volume of the angular sphere S^{8-p} as well. The resulting one-loop free energy is

$$I^{(1)}_{\rm WKB} \sim -(L\,T)^6 \int_1^\infty \frac{dx\sqrt{x}}{(1+xU_0\Delta)^r}\,. \tag{3.22}$$

Now, we need a B-field turned on in at least two planes ($r>1$), in order to achieve convergence at large U (this is reminiscent of the analogous condition to have a vanishing string coupling at infinity). In any case, the interpretation is not clear, because the standard UV/IR relation breaks down at the level of the formula for the Hawking temperature, since large energies (large U_0), correspond to low temperatures. In fact, the scaling at large temperature is that of a higher-than-seven-dimensional theory:

$$I^{(1)}_{\rm WKB} \sim -\frac{(L\,T)^6}{(U_0\Delta)^r} \sim -L^6\, a^{2r}\, T^{6+2r}\,. \tag{3.23}$$

Therefore, the cases $p = 5, 6$ continue to have non-standard features, although we do see a general tendency of the B-fields to make the large U behaviour less singular in all cases.

Other models. These WKB estimates can be extended to other interesting models. For example, we may consider NS5-branes of type-IIB and IIA related to D5-branes by a sequence of S- and T-dualities. The behaviour of $1/N^2$ corrections for all these models is essentially equivalent to that of type-IIB D5-branes, i.e. the commutative versions have semi-infinite cylinders that produce logarithmically divergent Hawking-radiation entropies [27, 17, 21]. On the other hand, turning on B-fields regulates this divergence and implies an effective quenching of $1/N$ corrections at large temperature. This is particularly clear for the case of type-IIB NS5-branes, whose metric is S-dual to that of D5-branes. Since this duality amounts to a conformal transformation of the metric, to which the optical metric is insensitive, we get the same physics of $1/N^2$ corrections: for large energy densities $\Delta U_0 \gg 1$,

$$\left[\frac{I_{\rm NC}}{I_{\rm free\ gas}}\right]^{(1)}_{\rm IIB\ NS5} \longrightarrow (U_0\Delta)^{-2r}\,. \tag{3.24}$$

Type-IIA NS5-branes can be obtained from type-IIB NS5-branes by a further T-duality along a commutative direction. We have a global factor of $\hat{f}^{-r/2}$ from the S-duality transformation from D5-branes to type-IIB NS5-branes. T-duality inverts this factor on one of the commuting coordinates. In addition, there are the usual factors of $\hat{f}$ for each NC coordinate. Thus, the optical volume integrand gets an additional factor of $\hat{f}^{r/2}$ in all, leading to

$$\left[\frac{I_{\rm NC}}{I_{\rm free\ gas}}\right]^{(1)}_{\rm IIA\ NS5} \longrightarrow (U_0\Delta)^{-3r}\,, \tag{3.25}$$

again in the large density limit. When the type-IIA D4-brane solution is lifted to eleven dimensions, one obtains a NC M5-brane model with $AdS_7 \times S^4$ geometry in the infrared region. The previous scaling gives now

$$\left[\frac{I_{\rm NC}}{I_{\rm C}}\right]^{(1)}_{\rm M5} \longrightarrow (U_0\Delta)^{-9r/2} \sim (a\,T)^{-9r}\,. \tag{3.26}$$

We remark that one interesting case was not discussed here in detail. It is the case in the presence of a nonvanishing "electric" NS-fields: $B_{0i} \neq 0$. Naively, one

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expects similar results to the purely "magnetic" case, at least as far as the arguments of ref. [14] concern. However, it was pointed out in ref. [16] that, at least in the particular case of $p = 3, r = 2$, the supergravity picture of the thermodynamics is fundamentally different at temperatures of the order of the timelike noncommutative scale.

Assuming that, for $B_{0i} \neq 0$, the dominant finite-temperature master field (the black hole) is also given by the substitutions (3.13) and (3.14) on the extremal solution, one finds that the behaviour described in [16] for $p = 3$ actually generalizes to all Dp-branes (this assumption might actually hide important subtleties, related to the proper treatment of the Wick rotation). The perturbative scale of "electric" nonlocality is $\sqrt{\theta_e} = 1/\sqrt{B_{0i}}$. At large 't Hooft coupling it develops into the length scales

$$\Delta_e = \left(\frac{\theta_e^2}{\lambda}\right)^{\frac{1}{7-p}}, \qquad a_e = \sqrt{\lambda\, \Delta_e^{5-p}}\,, \tag{3.27}$$

that characterize the supergravity solution. The temperature is related to the horizon radius U_0 by $T = T_{\rm C}(U_0)\sqrt{\hat{f}_e(U_0)}$, where $T_{\rm C}$ is the temperature of the commutative theory and $\hat{f}_e(U)$ is given by eq. (3.8) upon replacing Δ by Δ_e. This temperature/mass relation leads to negative specific heat for $U_0 \gg 1/\Delta_e$ and a maximum temperature of order $T_{\rm max} \sim 1/a_e$, a behaviour reminiscent of standard black-branes in asymptotically flat space, before the near-horizon scaling is taken as in eq. (3.1). In fact, the Einstein-frame metrics are exactly equal to the string-frame metrics of such asymptotically flat D-brane metrics (up to some rescalings of the coordinates,) precisely if $2r = p + 1$, i.e. when all the world-volume of the brane is noncommutative. This property was noticed in ref. [16] for the $p = 3, r = 2$ case. However, we see that the important qualitative features (a maximum temperature and a negative specific heat branch at large energy densities) generalize for arbitrary values of p and r, provided the *time* direction is noncommutative.

Having asymptotically flat regions at large U will surely complicate the workings of holography in these models. In particular, our WKB estimate for the one-loop entropy gives a *ten-dimensional* contribution, $S^{(1)} \sim T^9$, from supergravity modes at large $U \gg 1/\Delta_e$ in these models. Perhaps the theory imposes an effective cut-off of order $U_{\rm max} \sim 1/\Delta_e$ already at the planar level, as suggested by the existence of a maximum temperature at this scale. In this respect, it is interesting to notice that the branch with negative specific heat at large energy densities is dynamically suppressed in the canonical ensemble. The planar entropy in these models is given by $S = \hat{f}_e(U_0)^{-1/2}\, S_{\rm C}$, with $S_{\rm C}$ the entropy function of the commutative theory. Using the $T(U_0)$ function one finds that the entropy scales as $S \propto N^2\, T^{p-8}$ in the negative specific heat branch. From here one can get the free-energy excess over the vacuum:

$$(I - \beta\, E_{\rm vac})_{\rm planar} \longrightarrow +\frac{C_p}{7-p}\, \frac{N^2 L^p}{\lambda^{\frac{11-p}{2}}}\, \Delta^{-(p-7)^2/2}\, (\beta_{\rm NC})^{8-p} > 0\,, \tag{3.28}$$

with $C_p > 0$. It is *positive* in the region with negative specific heat. Therefore, there is an $O(e^{-N^2})$ suppression of this unstable branch in the canonical ensemble.

Another aspect of these solutions that we did not analyze in detail is the presence of light thermal winding modes at large radial coordinates. These cannot be eliminated through T-duality, and are sure to affect the physics at large values of U.

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References

[1] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rep.* **244** (1994) 77 [hep-th/9401139].

[2] S. Elitzur, A. Forge and E. Rabinovici, *Nucl. Phys.* **B 359** (1992) 131.

[3] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, *Nucl. Phys.* **B 438** (1995) 109 [hep-th/9410167];
E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124].

[4] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[5] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[6] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[7] A. Connes, *Noncommutative geometry*, Academic Press, 1994.

[8] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[9] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[10] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[11] M. Li, *Strings from IIB matrices*, *Nucl. Phys.* **B 499** (1997) 149 [hep-th/9612222].

[12] M. Li, *Comments on supersymmetric Yang-Mills theory on a noncommutative torus*, hep-th/9802052;
Y.-K.E. Cheung and M. Krogh, *Noncommutative geometry from 0-branes in a background B-field*, *Nucl. Phys.* **B 528** (1998) 185 [hep-th/9803031];
F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Mixed branes and M(atrix) theory on noncommutative torus*, hep-th/9803067; *Noncommutative geometry from strings and branes*, *J. High Energy Phys.* **02** (1999) 016 [hep-th/9810072]; *Dirac quantization of open strings and noncommutativity in branes*, hep-th/9906161;
C.-S. Chu and P.-M. Ho, *Constrained quantization of open string in background B-field and noncommutative D-brane*, hep-th/9906192;
V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[13] T. Filk, *Phys. Lett.* **B 376** (1996) 53:
T. Krajewski and R. Wulkenhaar, *Perturbative quantum gauge fields on the noncommutative torus*, hep-th/9903187;
the relevant combinatorics is actually isomorphic to that of twisted Eguchi–Kawai models, as in A. González-Arroyo and M. Okawa, *Phys. Rev.* **D 27** (1983) 2397;
see also H. Aoki et al., *Noncommutative Yang-Mills in IIB matrix model*, hep-th/9908141.

[14] D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, hep-th/9908056.

[15] A. Hashimoto and N. Itzhaki, *Non-commutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett.* **B 465** (1999) 142 [hep-th/9907166].

[16] J.M. Maldacena and J.G. Russo, *large-N limit of non-commutative gauge theories*, *J. High Energy Phys.* **09** (1999) 025 [hep-th/9908134].

[17] J.L.F. Barbón and E. Rabinovici, *Extensivity versus holography in anti-de Sitter spaces*, *Nucl. Phys.* **B 545** (1999) 371 [hep-th/9805143].

[18] G. 't Hooft, *Nucl. Phys.* **B 256** (1985) 727;
J.L.F. Barbón, *Horizon divergences of fields and strings in black hole backgrounds*, *Phys. Rev.* **D 50** (1994) 2712 [hep-th/9402004];
J.L.F. Barbón, *Remarks on thermal strings outside black holes*, *Phys. Lett.* **B 339** (1994) 41 [hep-th/9406209].

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[19] G.W. Gibbons and M.J. Perry, *Proc. Roy. Soc. Lond.* **A358** (1978) 467;
J.S. Dowker and G. Kennedy, *J. Phys.* **A11** (1978) 895.

[20] L. Susskind and J. Uglum, *Black hole entropy in canonical quantum gravity and superstring theory*, *Phys. Rev.* **D 50** (1994) 2700 [hep-th/9401070].

[21] J.L.F. Barbón, I.I. Kogan and E. Rabinovici, *On stringy thresholds in SYM/AdS thermodynamics*, *Nucl. Phys.* **B 544** (1999) 104 [hep-th/9809033].

[22] M. Li and Y.-S. Wu, *Holography and noncommutative Yang-Mills*, hep-th/9909085.

[23] M. Alishahiha, Y. Oz and M.M. Sheikh-Jabbari, *Supergravity and large-N noncommutative field theories*, *J. High Energy Phys.* **11** (1999) 007 [hep-th/9909215].

[24] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large-N limit of theories with sixteen supercharges*, *Phys. Rev.* **D 58** (1998) 046004 [hep-th/9802042].

[25] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, hep-th/9805114;
A.W. Peet and J. Polchinski, *UV/IR relations in AdS dynamics*, *Phys. Rev.* **D 59** (1999) 065011 [hep-th/9809022].

[26] G.T. Horowitz and J. Polchinski, *A correspondence principle for black holes and strings*, *Phys. Rev.* **D 55** (1997) 6189 [hep-th/9612146].

[27] J.M. Maldacena and A. Strominger, *Semiclassical decay of near-extremal fivebranes*, *J. High Energy Phys.* **12** (1997) 008 [hep-th/9710014].